# ALGEBRA OF FUZZY SUBGROUPS IN FUZZY SPACES

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***ABSTRACT:*** *In this pape,r we introduce some of the algebraic operations over fuzzy subgroups based on fuzzy spaces, namely union, intersection, direct sum and product of fuzzy subgroups based on fuzzy spaces.*

**Keyword** Fuzzy space, fuzzy subgroup, fuzzy normal subgroup, fuzzy direct product.

## 1. INTRODUCTION

In 1994 Dib[1] introduced the notion of fuzzy-group based on fuzzy spaces. Such an approach to the theory of fuzzy group was considered by many as a new formulation to the theory of fuzzy group and the theory of fuzzy mathematics in general. In his work, Dib remarked the absence of fuzzy universal set in the definition of fuzzy subgroup in the sense of Rosenfeld [2] and in the sense of Anthony-Sherwood [3] approaches. Thus to overcome the problems that occur in the construction of fuzzy groups, Dib introduce the notion of fuzzy space which play the role of a universal set in ordinary set theory. Using the notion of fuzzy space Dib was able to introduce the notion of fuzzy group in a similar way to the ordinary case, but using a fuzzy binary operation instead of the ordinary binary operation used by Rosenfeld, Anthony-Sherwood approaches.

Abdul Razak Salleh [4], Dib and Hassan[5] continue the work initiated in [1] and introduced the notion of normal fuzzy subgroup based on fuzzy spaces and obtained an interesting and important results regarding homomorphism and isomorphism in fuzzy group theory based on fuzzy space. In this paper, we introduce some operations regarding fuzzy subgroups such as union, intersection and direct sum based on fuzzy spaces and fuzzy binary operations introduced by Dib. Examples and some interesting results are also introduced and discussed.

## 2. BASIC FACTS ABOUT FUZZY SPACES AND FUZZY GROUPS

Throughout this article, we shall adopt the notations:

$X$ : for a non-empty set, $I$ : for the closed interval $[0,1]$ of real numbers.

The concept of *fuzzy space* $(X,I)$ was introduced and discussed by Dib [1], where $(X,I)$ is the set of all ordered pairs $(x,I);x\in X$ ; i.e., $(X,I)=\{(x,I):x\in X\}$, where $(x,I)=\{(x,r):r\in I\}$. The ordered pair $(x,I)$ is called a *fuzzy element* in the fuzzy space $(X,I)$. A *fuzzy subspace* $U$ of the fuzzy space $(X,I)$ is the collection of all ordered pairs $(x,u_x)$, where $x\in U_\circ$ for some $U_\circ\in X$ and $u_x$ is a subset of $I$, which contains at least one element beside the zero element. If it happens that $x\notin U_\circ$, then $u_x=0$.

An empty fuzzy subspace is defined as $\{(x,\phi_x):x\in\phi\}$. Let $U=\{(x,u_x):x\in U_\circ\}$ and $V=\{(x,v_x):x\in V_\circ\}$ be fuzzy subspaces of $(X,I)$. The *union* and *intersection* of $U$ and $V$ are defined respectively as follows:

$U\cup V=\{(x,u_x\cup v_x):x\in U_\circ\cup V_\circ\}$ and

$U\cap V=\{(x,u_x\cap v_x):x\in U_\circ\cap V_\circ\}$.

Clearly, both of $U\cup V$ and $U\cap V$ are fuzzy subspaces of the fuzzy space $(X,I)$. Let $(X,I)$ be a fuzzy space and let $A$ be a fuzzy subset of $X$ with $A_\circ$ denoted the support of the fuzzy subset $A$, i.e., $A_\circ=\{x:A(x)\neq 0\}$.

The fuzzy subset $A$ induces the following fuzzy subspaces of the fuzzy space $(X,I)$ :

- The lower fuzzy subspace $\underline{H}(A)=\{(x,[0,A(x)]):x\in A_\circ\}$.
- The upper fuzzy subspace $\overline{H}(A)=\{(x,\{0\}\cup[A(x),1]):x\in A_\circ\}$.
- The finite fuzzy subspace $H_\circ(A)=\{(x,\{0,A(x)\}):x\in A_\circ\}$.

Given two fuzzy spaces namely, $(X,I)$ and $(Y,I)$. A *fuzzy function* $\underline{F}$ from $(X,I)$ into $(Y,I)$ is defined as an ordered pair $\underline{F}=(F,\{f_x\}_{x\in X})$, where $F$ is a function from $X$ into $Y$, and $\{f_x\}_{x\in X}$ is a family of onto functions (called *co-membership functions*) $f_x:I\to I$, satisfying the conditions:

- $f_x$ is non decreasing on $I$ ,
- $f_x(0)=0$ , $f_x(1)=1$.

A *fuzzy binary operation* $\underline{F}=(F,f_x)$ on the fuzzy space $(X,I)$ is a fuzzy function from $(X,I)\bigotimes(X,I)\to(X,I)$, where $F:X\times X\to X$ with onto co-membership functions $f_{xy}:I\bigotimes I\to I$ which satisfies $f_{xy}(r,s)\neq 0$ if $r\neq 0$ and $s\neq 0$, where $I\bigotimes I$ is the vector lattice with partial order defined for all $r_1,r_2,s_1,s_2\in I$ by

- $(r_1,r_2)\le(s_1,s_2)$ if and only if $r_1\le s_1$ and $r_2\le s_2$ whenever $s_1\neq 0$ and $s_2\neq 0$.
- $(0,0)\le(s_1,s_2)$ whenever $s_1=0$ or $s_2=0$.

The fuzzy binary operation $\underline{F}=(F,f_x)$ on $(X,I)$ is said to be *uniform* if the associated co-membership functions $f_{xy}$ are identical for all $x,y\in X$ , i.e., $f_{xy}=f$ for all $x,y\in X$ . A fuzzy space $(X,I)$ together with a fuzzy binary operation $\underline{F}=(F,f_x)$ is said to be a *fuzzy groupoid* and is denoted by $((X,I);\underline{F})$. A *fuzzy semigroup* is a fuzzy groupoid which is associative. A *fuzzy monoid* is a fuzzy semigroup admits an identity $(e,I)$ such that for every $(x,I)\in(X,I)$ we have

$(x,I)\underline{F}(e,I)=(e,I)\underline{F}(x,I)=(x,I)$. A *fuzzy group* is a fuzzy monoid in which each fuzzy element $(x,I)$ has an inverse $(x,I)^{-1}=(x^{-1},I)$ such that $(x,I)\underline{F}(x,I)^{-1}=(x,I)^{-1}\underline{F}(x,I)=(e,I)$. A fuzzy group is said to be *abelian (commutative)* if and only if for any $(x,I),(y,I)\in(X,I)$ we have $(x,I)\underline{F}(y,I)=(y,I)\underline{F}(x,I)$.

If $U$ is a fuzzy subspace of the fuzzy space $(X,I)$ then $(U;F)$ is a fuzzy subgroup of the fuzzy group $((X,I);F)$ if $(U;F)$ itself defines a fuzzy group. The next theorem obtained by Dib [4] gives a necessary and sufficient conditions for fuzzy subgroups:

**Theorem 2.1** *A fuzzy subspace* $U=\{(x,u_x):x\in U_\circ\}$ *of the fuzzy space* $(X,I)$ *is a fuzzy subgroup of the fuzzy group* $((X,I);\underline{F})$ *under the fuzzy binary operation* $\underline{F}=(F,f_{xy})$ *iff*

- $(U,F)$ is an ordinary subgroup of the ordinary group $(X,F)$.
- $f_{xy}\left(A(x),A(y)\right)=A(xFy)$

If $U=\{(x,u_x):x\in U_\circ$ is a fuzzy subgroup of the fuzzy group $((X,I);\underline{F})$, then for every fuzzy element $(x,I)$ of $(X,I)$, the fuzzy subspace define by

$$(x,I)U=(x,I)\underline{F}U=\{x\underline{F}z,f_{xz}(I,u_z)\}$$

is called a *left coset* of the fuzzy subgroup $U$.

## 3. ALGEBRA OF FUZZY SUBGROUPS

In this section we define the operations of union, intersection, external and internal direct product of fuzzy subgroups based on fuzzy subspaces. Before introducing these operations we will introduce the notion of *cyclic fuzzy* subgroups based on fuzzy spaces.

Consider the fuzzy group $((X,I);F)$ and the fuzzy element $(a,I)$ in $((X,I);F)$, by $(a^2,I)$ we mean $(a,I)F(a,I)$. In general

$$(a^n,I)=\underbrace{(a,I)F(a,I)\cdots F(a,I)}_{\text{n−times}}$$

The next theorem gives an analoguous result to the classical case regarding fuzzy subgroups generated by a fuzzy element.

**Theorem 3.1** *Let* $((X,I);F)$ *be a fuzzy group and let* $(a,I)\in((X,I);F)$. *Then* $(H;\underline{F})=\{(a^n,I):n\in \mathrm{Z}\}$ *is a fuzzy subgroup of the fuzzy group* $((X,I);F)$. *Moreover* $(H;\underline{F})$ *is the smallest fuzzy subgroup containing the fuzzy element* $(a,I)$.

*Proof.* We will make use of Theorem 2.1 to prove our result. From classical abstract algebra we have $(H;\underline{F})$ is an ordinary subgroup of the ordinary group $(X,F)$. Also since $f_{xy}$ are onto co-membership functions then $f_{xy}(I,I)=f_{xy}(I)$. That is $f_{xy}\left(A(x);A(y)\right)=A(xFy)$ for all $x,y\in X$.

Based on the above theorem we have the following definition.

**Definition 3.2** *Let* $((X,I);F)$ *be a fuzzy group and let* $(a,I)\in((X,I);F)$. *Then the fuzzy subgroup* $(H;\underline{F})=\{(a^n,I):n\in \mathrm{Z}\}$ *is called a cyclic fuzzy subgroup generated by the fuzzy element* $(a,I)$ *and is denoted by* $\langle(a,I)\rangle$. *The fuzzy element* $(a,I)$ *is called a generator of the fuzzy group* $((X,I);F)$ *if* $\langle(a,I)\rangle=((X,I);F)$. *A fuzzy group* $((X,I);F)$ *is called cyclic fuzzy group if there exists a fuzzy element* $(a,I)\in((X,I);F)$ *that generates* $((X,I);F)$.

**Example 3.3** *Consider the set of integers* $\mathrm{Z}$. *Define the fuzzy binary operation* $\underline{F}=\{F;f_{xy}\}$ *over the fuzzy space* $(\mathrm{Z},I)$ *as follow:*

$F(x,y)=x+y$. That is, $F$ denotes the ordinary addition of integers.

$f_{xy}(r,s)=f(r,s)=r\vee s$. That is, $f$ denotes the maximum of membership values. Thus $((\mathrm{Z},I);\underline{F})$ defines a fuzzy group. Now consider the fuzzy elements $(1,I)$ and $(-1,I)$. Clearly both of $(1,I)$ and $(-1,I)$ generates the fuzzy group $((\mathrm{Z},I);\underline{F})$. That is $\langle(1,I)\rangle=((\mathrm{Z},I);\underline{F})=(-1,I)$.

**Definition 3.4** *The union of two fuzzy subgroups* $(U;\underline{F}):U=\{(x;u_x):x\in U_\circ\}$ *and* $(V;\underline{F}):V=\{(y;v_y):y\in V_\circ\}$ *of the fuzzy group* $((X,I);F)$ *is the fuzzy subset given by* $(U;\underline{F})\cup(V;\underline{F})=\{z:z\in U\vee z\in V\}$.

**Remark 3.1** *Note that the relation* $\vee$ *(or) in the above definition is a classical or while the fuzzy s-norm (fuzzy or) acts on the membership values assigned to each element* $x$ *by the fuzzy co-membership functions which can be any s-norm.*

The next theorem characterize the notion of union of fuzzy subgroups.

**Theorem 3.5** *Suppose that* $((X,I);F)$ *is a fuzzy group and* $(U;\underline{F}),(V;\underline{F})$ *are fuzzy subgroups of* $((X,I);F)$. *Then the following are true:*

*(1) Any fuzzy subgroup contained in* $(U;\underline{F})\cup(V;\underline{F})$ *is contained either in* $(U;\underline{F})$ *or in* $(V;\underline{F})$.

*(2) If* $(U;\underline{F})\cup(V;\underline{F})$ *is a fuzzy subgroup then either* $(U;\underline{F})$ *is contained in* $(V;\underline{F})$ *or* $(V;\underline{F})$ *is contained in* $(U;\underline{F})$.

*Proof.* Let $((X,I);F)$ be a fuzzy group with fuzzy subgroups $(U;\underline{F}),(V;\underline{F})$ and $(L;\underline{F})$ such that $(L;\underline{F})\subset(U;\underline{F})\cup(V;\underline{F})$. Now suppose that $(L;\underline{F})$ is contained neither in $(U;\underline{F})$ nor in $(V;\underline{F})$, then we can find fuzzy elements $(h,L_h)$ and $(k,L_k)$ such that

$(h,L_h)\in(L;\underline{F})-(U;\underline{F})$ and $(k,L_k)\in(L;\underline{F})-(V;\underline{F})$ Now consider the fuzzy elements $(h,L_h),(k,L_k)$ and $(hk,L_{hk})$, where $hk=h\underline{F}k$. Since $(hk,L_{hk})\in(L;\underline{F})$ so $(hk,L_{hk})\in(U;\underline{F})$ or $(hk,L_{hk})\in(V;\underline{F})$. If $(hk,L_{hk})\in(U;\underline{F})$ then clearly since $(k;L_k)\in(U;\underline{F})$ then $(h,L_h)$ must be in $(U;\underline{F})$ which is a contradiction. Similarly, if $(hk,L_{hk})\in(V;\underline{F})$ then clearly since $(h;L_h)\in(V;\underline{F})$ then $(k,L_k)$ must be in $(U;\underline{F})$ which is also a contradiction. Thus by contradiction we have either $(L;\underline{F})\le(U;\underline{F})$ or $(L;\underline{F})\le(V;\underline{F})$.

For the second part of the theorem, observe that if $(U;\underline{F})\cup(V;\underline{F})$ is a fuzzy subgroup we can choose $(L;\underline{F})=(U;\underline{F})\cup(V;\underline{F})$ to get $(U;\underline{F})\cup(V;\underline{F})\le(U;\underline{F})$ or $(U;\underline{F})\cup(V;\underline{F})\le(V;\underline{F})$ yielding $(U;\underline{F})\le(V;\underline{F})$ or $(V;\underline{F})\le(U;\underline{F})$.

**Definition 3.6** *The intersection of two fuzzy subgroups* $(U;\underline{F}):U=\{(x;u_x):x\in U_\circ\}$ *and* $(V;\underline{F}):V=\{(y;v_y):y\in V_\circ\}$ *of the fuzzy group* $((X,I);\underline{F})$ *is the fuzzy subset given by* $(U;\underline{F})\cap(V;\underline{F})=\{z:z\in U\wedge z\in V\}$.

**Remark 3.2** *The relation* $\wedge$ *(and) in the above definition is a classical and while the fuzzy t-norm (fuzzy and) acts on the membership values assigned to each element* $x$ *by the fuzzy co-membership functions which can be any t-norm.*

**Theorem 3.7** *The intersection of two fuzzy subgroups* $(U;\underline{F})$ *and* $(V;\underline{F})\}$ *of a fuzzy group* $((X,I);F)$ *is a fuzzy subgroup.*

*Proof.* The proof is straightforward.

**Definition 3.8** *Let* $((X_1,I);\underline{F}),((X_2;I);\underline{H})$ *be two fuzzy groups. The external direct product of* $((X_1,I);\underline{F}),((X_2;I);\underline{H})$ *written as* $((X_1,I);\underline{F})\oplus((X_2;I);\underline{H})$ *is the set of ordered pairs* $\{((x_1,I),(x_2,I)):(x_i,I)\in(X_i,I)\}$.

If we define the fuzzy binary operation $\underline{G}=\{G;g_{xy}\}$ over the fuzzy space $((X_1,I);\underline{F})\oplus((X_2;I);\underline{H})$ such that for any fuzzy elements $((x_1,I),(x_2,I))$ and $((y_1,I),(y_2,I))\in((X_1,I);\underline{F})\oplus((X_2;I);\underline{H})$

$$((x_1,I),(x_2,I))\underline{G}((y_1,I),(y_2,I))=((x_1,I)\underline{F}(y_1,I)(x_2,I)\underline{H}(y_2,I))$$

Then, clearly $((X_1,I);\underline{F})\oplus((X_2;I);\underline{H})$ defines a fuzzy group. Note that the fuzzy binary operation $\underline{G}$ was defined in terms of the fuzzy binary operations $\underline{F}$ and $\underline{H}$.

The above definition can be generalized to a family of fuzzy groups. That is given a family of fuzzy groups $((X_1,I);\underline{F}_1);((X_2,I);\underline{F}_2)\cdots((X_n,I);\underline{F}_n)$ the external direct product is the set of n-tuples for which the $i$-th component is a fuzzy elements in $((X_i,I);\underline{F}_i)$. Symbolically

$$((X_1,I);\underline{F}_1)\oplus((X_2,I)\oplus\underline{F}_2)\cdots\oplus((X_n,I);\underline{F}_n)$$

such that $(x_i,I)\in((X_i,I);\underline{F}_i)$.

A major difference between classical groups and fuzzy groups based on fuzzy spaces that elements of the fuzzy group $((X,I);\underline{F})$ and elements in a fuzzy subgroup $(U;\underline{F})$ need not be associative, i.e., $(a\underline{F}b)\underline{F}c\ne a\underline{F}(b\underline{F}c)$. where $a,b$ and $c$ are any fuzzy elements of $U$ or $(x,I)$ such that one or two of $a,b$ or $c$ belong to $U$.

Based on the above, a fuzzy subgroup $(U;\underline{F})$ of a fuzzy group $((X,I);\underline{F})$ is said to be *associative fuzzy subgroup* if associativity holds between fuzzy elements of $U$ and fuzzy elements of $((X,I);\underline{F})$. Similarly, a fuzzy subgroup $(U;\underline{F})$ of a fuzzy group $((X,I);\underline{F})$ is said to be *normal fuzzy subgroup* if the following satisfied:

(1) $(U;\underline{F})$ is an associative fuzzy subgroup.

(2) $(x,I)U=U(x,I),x\in X$.

Now we are ready to define a fuzzy group in terms of fuzzy subgroups using internal product.

**Definition 3.9** *Let* $((X,I);\underline{F})$ *be a fuzzy subgroup and let* $(U_1;\underline{F}),(U_2;\underline{F}),\cdots(U_n;\underline{F})$ *be fuzzy subgroups of* $((X,I);\underline{F})$. *We say that* $((X,I);\underline{F})$ *is the (internal) fuzzy direct product of the fuzzy subgroups* $U_i$ *if:*

$H_i$ is a fuzzy normal subroups of $((X,I);\underline{F})$ for all $i$;

$((X,I);\underline{F})=(U_1;\underline{F})(U_2;\underline{F})\cdots(U_n;\underline{F})$;

Each distinct pair of fuzzy subgroups has the trivial intersection, namely the fuzzy identity element.

If the fuzzy group $((X,I);\underline{F})$ is the internal direct product of the fuzzy subgroups $(U;\underline{F}),(V;\underline{F})$ then we write $((X,I);\underline{F})=(U;\underline{F})\otimes(V;\underline{F})$.

The next theorem highlights the relation between internal and external direct products

**Theorem 3.10** *Let* $((X_1,I);\underline{F}),((X_2;I);\underline{H})$ *be two fuzzy groups with eternal direct sum* $((X_1,I);\underline{F})\oplus((X_2;I);\underline{H})$. *Define* $((\overline{X}_1,I);\underline{F})$ *and* $((\overline{X}_2;I);\underline{H})$ *as:*

$$((\overline{X}_1,I);\underline{F})=\big((x_1,I),(e_2,I):x_1\in X_1,e_2 isthefuzzyidentintyelemntof\ X_2\big)$$

$$((\overline{X}_2,I);\underline{F})=\big((e_1,I),(x_2,I):x_2\in X_2,e_1 isthefuzzyidentintyelemntof\ X_1\big)$$

*Then* $((\overline{X}_1,I);\underline{F})$ *and* $((\overline{X}_2,I);\underline{F})$ *are fuzzy subgroups of* $((X_1,I);\underline{F})\oplus((X_2;I);\underline{H})$. *Moreover, the external direct product* $((X_1,I);\underline{F})\oplus((X_2;I);\underline{H})$ *is the internal direct product of* $((\overline{X}_1,I);\underline{F})$ *and* $((\overline{X}_2,I);\underline{F})$.

## 4.CONCLUSION

Fuzzy group theory play a fundamental rule in fuzzy mathematics, but in the absence of the notion of fuzzy universal sets, formulation of the intrinsic definition for fuzzy subgroup is not evident. In this paper we continue the study of fuzzy groups based on fuzzy universal sets by introducing some fuzzy algebraic operations over fuzzy groups in order to build new fuzzy group from a given fuzzy groups. The use of fuzzy space as a universal fuzzy sets corrects the deviation in the theory of fuzzy groups.

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